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Procedia Engineering 75 (2014) 164 – 168
Procedia Engineering
www.elsevier.com/locate/procedia

MRS Singapore - ICMAT Symposia Proceedings
International Conference on Materials for Advanced Technologies 2013

Free Vibration of Exponential Functionally Graded Beams with Single Delamination

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Abstract

FGMs are regarded as one of the most promising candidates for future advanced composites in many engineering sectors, but they may suffer the problem of delamination. Delaminations in structures may significantly reduce the stiffness and strength of the structure and may affect their vibration characteristics. In the present study, an analytical solution is developed to study the free vibration of exponential functionally graded beams with a single delamination. Kirchhoff-Love hypothesis, the 'free mode' and 'constrained mode' assumption in delamination vibration are adopted. The shifting of neutral axis due to asymmetrical distribution of material property (in thickness direction) is also taken into consideration. This is the first study on the influences of delamination (its length and location) on the vibration of exponential functionally graded beams. Results show that the natural frequency increases as the Young's modulus ratio increases, but such increase is smaller when the beam suffers a longer delamination. Furthermore, the effect of delamination length and longitudinal location on reducing natural frequency is aggravated when the material property (Young's modulus and density) changes less dramatically from the bottom to the top. The difference of natural frequency between 'free mode' and 'constrained mode' becomes smaller with a decreasing Young's modulus ratio. The analytical results of this study can serve as the benchmark for FEM and other numerical solutions.

Keywords: vibration; delamination; analytical modeling; functionally graded material

Nomenclature

u_i	axial displacement of beam i
w_i	bending displacement of beam i
ω	the natural frequency of the delaminated FGM beam

1. Introduction

Functionally graded materials (FGMs) are microscopically inhomogeneous composites that are usually made from a mixture of metals and ceramics. FGMs are regarded as one of the most promising candidates for future advanced composites in many engineering sectors such as the aerospace, aircraft, automobile, and defence industries, and most recently the electronics and biomedical sectors [1].

Atmane et al. [2] studied the free vibration behaviour of exponential functionally graded beams with varying cross-section. Yang and Chen [3] studied the analytical solution to the free vibration and buckling of functionally graded beams with edge cracks. Literature review shows that although there are quite a few papers presenting crack and fracture analyses of FGM structures, no work investigating the vibration behaviour of delaminated FGM structures has been reported. One major failure

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doi:10.1016/j.proeng.2013.11.041

mode of FGMs is cracking perpendicular to the material gradient direction which corresponds to delamination as real FGMs are usually multi-layered materials [4] and the crack grows along the interface. Delamination also occurs as one of four distinctive stages of the damage development of functionally graded thermal barrier coatings [5]. The buckling driven delamination problem under mechanical and thermal loadings was investigated [6] by using analytical and finite element methods.

In the present study, an analytical solution is developed to study the free vibration of exponential functionally graded beams with a single delamination. Kirchhoff-Love hypothesis, the 'free mode' and 'constrained mode' assumption in delamination vibration are adopted. This is the first study on the effects of delamination (its length and location) on the natural frequency of exponentially functionally graded beams. The analytical results of this study can serve as the benchmark for FEM and other numerical solutions.

2. Formulations

2.1. Governing equations and analytical solutions

As is shown in Fig. 1 (a), consider a FGM beam of length L and thickness H_1, with a single delamination of length a, located at a distance d_a from the center of the beam. The beam can be subdivided into three span-wise regions, one delaminated region and two undelaminated region. The delamination region is comprised of two segments (delaminated layers), beam 2 and 3, which are joined at their ends to the integral segments, beam 1 and 4. The Young's modulus $E(z)$ and mass density $\rho(z)$ of the beam vary in the thickness direction (z) only and follow the exponential distributions shown below

$$E(z) = E_0 e^{\beta z},\ \rho(z) = \rho_0 e^{\beta z} \tag{1}$$

where E_0 and ρ_0 are the Young's modulus and density of the FGM beam at the midplane.

Based on the Kirchhoff-Love hypothesis, $\bar{u}_i(x, z_i, t)$ and $\bar{w}_i(x, z_i, t)$, which refer to the displacements parallel to the x- and z-axes of an arbitrary point in beam i respectively (as is shown in Fig. 1 (b)), are

$$\bar{u}_i(x, z, t) = u_i(x, t) - z_i\, \partial w_i/\partial x \tag{2}$$

$$\bar{w}_i(x, z, t) = w_i(x, t) \tag{3}$$

where $u_i(x, t)$ and $w_i(x, t)$ are the axial displacement and flexural displacement of the midplane of beam i. The governing equations for beam i , with axial inertia term being neglected, can be derived as follows:

$$A_{11}(i)\, \partial^2 u_i/\partial x^2 - B_{11}(i)\, \partial^3 w_i/\partial x^3 = 0 \tag{4}$$

$$(D_{11}(i) - B_{11}^2(i)/A_{11}(i))\, \partial^4 w_i/\partial x^4 + I_{1,i}\, \partial^2 w_i/\partial t^2 = 0 \tag{5}$$

where $I_{1,i} = \int_{-\frac{H_i}{2}}^{\frac{H_i}{2}} \rho(z)\, dz_i$. (6)

For free harmonic vibration, the deformation mode shape can be assumed as

$$u_i(x, t) = U_i(x) e^{i\omega t} \tag{7}$$

$$w_i(x, t) = W_i(x) e^{i\omega t} \tag{8}$$

where ω is the natural frequency of the delaminated FGM beam. Substituting Eqs (7) and (8) into Eqs (4) and (5), one can have the solution One can have the solution as:

$$W_i(x) = e_{i,1} \sin(\lambda x) + e_{i,2} \cos(\lambda x) + e_{i,3} \sinh(\lambda x) + e_{i,4} \cosh(\lambda x) \tag{9}$$

$$U_i(x, t) = \lambda_i\, B_{11}(i)/A_{11}(i)\, [e_{i,1} \cos(\lambda x) - e_{i,2} \sin(\lambda x) + e_{i,3} \cosh(\lambda x) + e_{i,4} \sinh(\lambda x)] + g_i x + g_{i0} \tag{10}$$

where $\lambda_i^4 = \omega^2 I_{1,i} / d_i$ and $d_i = D_{11}(i) - B_{11}^2(i)/A_{11}(i)$ (11)

2.2. 'Free mode'

Governing equations 4 and 5 are applied to the four interconnected sub-beams, respectively (Fig. 1(b)). The appropriate boundary conditions that can be applied at the supports, $x = x_1$ and $x = x_4$, are $W_i = 0$, $W_i' = 0$ and $U_i = 0$ if the end of i the beam is clamped, $W_i = 0$, $W_i'' = 0$ and $U_i = 0$ if hinged, $W_i'' = 0$, $W_i''' = 0$ and $U_i' = 0$ if free, where i=1, 4.

The continuity conditions for flexural displacement, flexural slope, axial force and shear force at are:

$$W_1 = W_2 = W_3 \tag{12}$$

$$W_1' = W_2' = W_3' \tag{13}$$

$$N_1 = N_2 + N_3 \tag{14}$$

$$Q_1 = Q_2 + Q_3 \tag{15}$$

where N_i and Q_i are the axial force and shear force of beam i, respectively.

The continuity conditions for axial displacement [7], when using the neutral axis as the referencing axis, as is shown in Fig. 1(c), at $x = x_2$ are:

$$U_1 - (H_3/2 + D_2)\, \partial W_1/\partial x = U_2 - D_2\, \partial W_2/\partial x \tag{16}$$

$$U_1 + (H_2/2 - D_3)\, \partial W_1/\partial x = U_3 - D_3\, \partial W_3/\partial x \tag{17}$$

where D_i is distance between the neutral axis and the midplane of beam i.

The continuity condition for bending moment at $x = x_2$ is:

$$\mathrm{M}_1^{'} = \mathrm{M}_2^{'} + \mathrm{M}_3^{'}\text{-}\mathrm{N}_2 \times \mathrm{D}_1^2 + \mathrm{N}_3 \times \mathrm{D}_1^3 \tag{18}$$

where D_1^2 and D_1^3 are the distances between the two neutral axes of beam 1 and beam 2, beam 1 and beam 3 respectively, and M_i' is the equivalent bending moment when using the neutral axis as the referencing axis, instead of the midplane, therefore $\mathrm{M}_i^{'} = \mathrm{M}_i + \mathrm{N}_i \cdot \mathrm{D}_i$.

And similarly, one can derive the continuity conditions at $x = x_3$.

2.3. 'Constrained mode'

In 'constrained mode' model, beam 2 and 3 vibrates together, i.e. $w_2 = w_3 = w_{\mathrm{II}}$. The bottom of beam 2 and top of beam 3 have the same axial displacement, therefore

$$u_2 + \frac{H_2}{2}\frac{\partial w_{\mathrm{II}}}{\partial x} = u_3 - \frac{H_3}{2}\frac{\partial w_{\mathrm{II}}}{\partial x} \tag{19}$$

For the 'constrained mode' model, the governing equations for beam 1 and beam 4 are identical to Eqs. (8) and (9). For beam 2 and 3, i.e. section II, the governing equations are

$$A_{11}(2)\frac{\partial^2 u_2}{\partial x^2} + A_{11}(3)\frac{\partial^2 u_3}{\partial x^2} - (B_{11}(2) + B_{11}(3))\frac{\partial^4 w_{\mathrm{II}}}{\partial x^2} = 0 \tag{20}$$

$$\left(D_{11}(2) + D_{11}(3) - \frac{B_{11}^2(2)A_{11}(3)+B_{11}^2(3)A_{11}(2)}{A_{11}(2)A_{11}(3)}\right)\frac{\partial^4 w_{\mathrm{II}}}{\partial x^4} + (I_{1,2} + I_{1,3})W_{\mathrm{II}} = 0 \tag{21}$$

Substituting Eq. (23) to Eqs. (24) and (25), therefore

$$\left(A_{11}(2) + A_{11}(3)\right)\partial^2 u_2/\partial x^2 - (B_{11}(2) + B_{11}(3) - H_1/2\, A_{11}(3))\,\partial^4 w_{\mathrm{II}}/\partial x^2 = 0 \tag{22}$$

The boundary conditions for the 'constrained mode' are identical to that of the 'free mode'. The continuity conditions for flexural displacement and slope, axial displacement, shear force and bending moments at $x = x_2$ are:

$$W_I = W_{II} \tag{23}$$

$$W_{\mathrm{I}}' = W_{\mathrm{II}}' \tag{24}$$

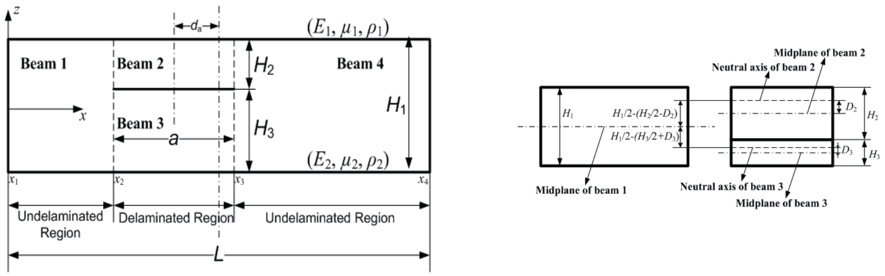
$$\mathrm{U}_{\mathrm{I}}\text{-}(\frac{\mathrm{H}_3}{2} + \mathrm{D}_2)\frac{\partial \mathrm{W}_{\mathrm{I}}}{\partial \mathrm{x}} = \mathrm{U}_2\text{-}\mathrm{D}_2\frac{\partial \mathrm{W}_{\mathrm{II}}}{\partial \mathrm{x}} \tag{25}$$

$$N_{\mathrm{I}} = N_{\mathrm{II}} = N_2 + N_3 \tag{26}$$

$$Q_{\mathrm{I}} = Q_{\mathrm{II}} = Q_2 + Q_3 \tag{27}$$

$$\mathrm{M}_{\mathrm{I}} = \mathrm{M}_1{}' = \mathrm{M}_{\mathrm{II}} = \mathrm{M}_2^{'} + \mathrm{M}_3^{'}\text{-}\mathrm{N}_2 \times \mathrm{D}_1^2 + \mathrm{N}_3 \times \mathrm{D}_1^3 \tag{28}$$

And similarly, one can derive the continuity conditions at $x = x_3$. The total number of boundary and continuity conditions is 24 for 'free mode' and 18 for 'constrained mode', which is equal to the total number of unknown coefficients. A non-trivial solution exists only when the determinant of the coefficient matrix vanishes. The natural frequencies can be obtained as eigenvalues.



(a) (b)

Fig. 1. (a) An exponential functionally graded beam with a single delamination; (b) Axial displacement continuity condition.

3. Results and Discussions

3.1. Verification

Table 1 shows the non-dimensional first natural frequency λ^2 of an isotropic beam (by denoting E_2 / E_1= 0.999) with a single of central midplane delamination of various lengths, compared with the analytical results of Liu and Shu [7] and FEM results of Lee [8]. As is shown in Table 1, the results of the current study agree well with previous published results.

Table 2 shows the first three normalized natural frequencies ($\Omega_n = \omega_n / \sqrt{d_{10}}$) of FGM beams, with clamped-free and hinged-hinged boundary conditions, where d_{10} is the value of d_1 of an isotropic beam (E_2 / E_1 = 1.0) [2]. This example was previously analyzed by Atmane et al. [2], Yang and Chen [3] and Yang et al. [9]. The natural frequencies obtained in the present study are in good agreement with previous results.

Table 1.Non-dimensional fundamental frequency (λ^2) of a clamped-clamped isotropic beam with a midplane delamination.

Delamination length, *a/L*	Present Constrained & Free	Analytical Liu and Shu [7]	FEM Lee [8]
0.10	22.37	22.37	22.36
0.20	22.36	22.35	22.35
0.30	22.24	22.23	22.23
0.40	21.83	21.83	21.82

Table 2. First three normalized natural frequencies $\Omega_n = \omega_n / \sqrt{d_{10}}$ of undelaminated exponential FGM beams, with hinged-hinged boundary condition.

E_2 / E_1	Mode Number	Present Free	Yang and Chen [3] Yang et al. [9]	Atmane et al. [2]
0.2	1	9.27	9.27	9.272
	2	37.09	37.09	37.090
	3	83.28	83.28	83.453

3.2. FGM beams suffering a single delamination

Firstly, the first normalized natural frequency ($\Omega = \omega / \sqrt{d_{10}}$) FGM beam with a single delamination is studied. The top layer is made Zirconia (E_1 = 200GPa) and the bottom layer is made of aluminum (E_2 = 70GPa). The FGM beam then suffers a delamination with varying length and locations, with two different boundary condition considered.

It is assumed in the following study that the FGM beam's thickness H_1 = 0.1m; the slenderness ratio L / h = 10; the Young's modulus ratio E_2 / E_1 varies from 0.01 to 1.0, where E_1 and E_2 denote the Young's modulus at the top and bottom surfaces of the beam respectively. Note that E_2 / E_1 = 1.0 is a special case where the beam is isotropic homogeneous. The top surface of the beam is 100% aluminum with the material parameters: E_1 = 70GPa, ρ_1=2780kg/m^3, μ_1= 0.33, as is studied by Atmane [2].

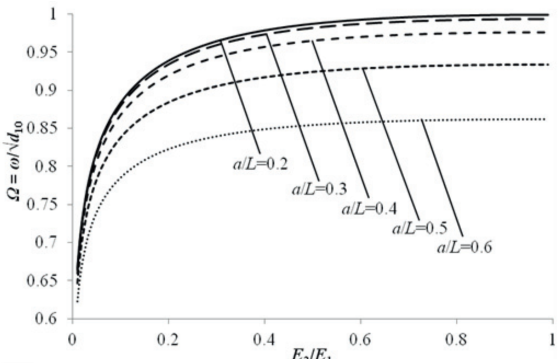


Fig. 2. 'Free mode' normalized natural frequencies versus the Young's modulus ratio with various delamination lengths, d_a=0.0 and H_2= 0.5, with a clamped-clamped boundary condition.

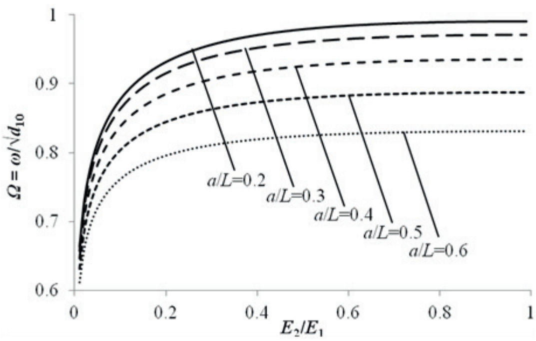


Fig. 3. 'Free mode' normalized natural frequencies versus the Young's modulus ratio with various delamination lengths, d_a=0.0 and H_2= 0.5, with a clamped-free boundary condition.

Fig. 2 shows the 'free mode' normalized natural frequencies versus the Young's modulus ratio E_2 / E_1 with various central midplane (d_a=0.0, H_2= 0.5) delamination lengths a / L = 0.2, 0.3, 0.4, 0.5, and 0.6, with a clamped-clamped boundary condition. It can be observed from Fig. 2 that the natural frequencies of FGM beams increase as the Young's modulus ratio increases, till the beam becomes isotropic. Such increase is smaller when the beam suffers a longer delamination. The natural frequencies decrease as the delamination grows longer. The differences of natural frequencies between delamination length a / L = 0.2 and 0.3 (($\Omega_{a/L=0.2}$ - $\Omega_{a/L=0.3}$) / $\Omega_{a/L=0.2}$), 0.3 and 0.4, 0.4 and 0.5 as well as 0.5 and 0.6, becomes bigger with a bigger Young's modulus ratio. It indicates that the effect of delamination length on reducing natural frequency is aggravated when the material property (E_2/E_1) changes less dramatically from the bottom to the top.

The 'constrained mode' natural frequencies of the same beam in Fig. 2 are shown in Fig. 4. Similar results can be obtained on how the Young's modulus ratio aggravates the effect of delamination. When compare the results between Fig. 2 and Fig. 4, the 'constrained mode' natural frequency is bigger than the 'free mode' and such difference becomes smaller with a decreasing Young's modulus ratio.

Fig. 5 shows the ‘free mode’ normalized first natural frequencies versus the Young’s modulus ratio E_2/E_1 with various midplane delamination (H_2 = 0.5 and a / L=0.2) longitudinal locations d_a , with a clamped-clamped boundary condition. The effect of delamination longitudinal location on reducing the natural frequency is aggravated when the Young’s modulus ratio increases.

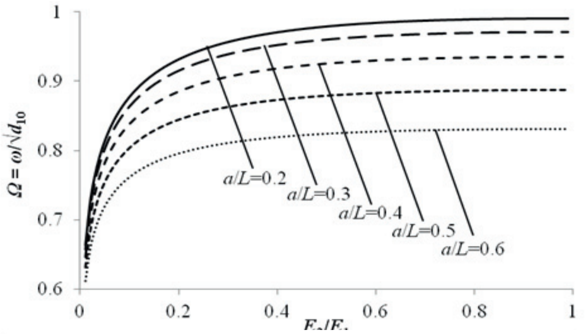


Fig. 4. ‘Constrained mode’ normalized natural frequencies versus the Young’s modulus ratio with various delamination lengths, d_a=0.0 and H_2= 0.5, with a clamped-clamped boundary condition.

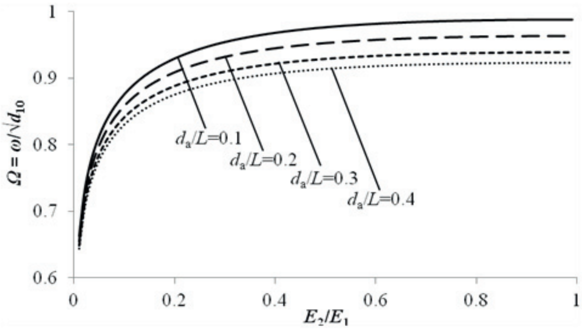


Fig. 5. ‘Free mode’ normalized natural frequencies versus the Young’s modulus ratio with various delamination locations, a / L=0.2 and H_2= 0.5, with a clamped-clamped boundary

4. Conclusions

In the present study, an analytical solution is developed to study the free vibration of exponential functionally graded beams with a single delamination. Kirchhoff-Love hypothesis, the ‘free mode’ and ‘constrained mode’ assumption in delamination vibration are adopted. This is the first study on the influences of delamination (its length and location) on the natural frequency of exponentially functionally graded beams. Based on the theoretical investigations the following conclusions may be drawn.

1. The ‘free mode’ natural frequencies increase as the Young’s modulus ratio increases, but such increase is smaller when the beam suffers a longer delamination
2. The effect of delamination length on reducing natural frequency is aggravated when the material property (E_2/E_1) changes less dramatically from the bottom to the top.
3. The difference of natural frequency between ‘free mode’ and ‘constrained mode’ becomes smaller with a decreasing Young’s modulus ratio.
4. The effect of delamination longitudinal location on reducing the natural frequency is aggravated when the Young’s modulus ratio increases, till the beam becomes isotropic.

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